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THE CONCEPT OF THE INFINITE.

EVERY student of the deeper problems of theology is familiar with what is often known as "the problem of the Infinite." Under the meaning of this one phrase may be brought a number of distinguishable, but closely connected questions. Some of these questions appear, in a more or less veiled form, even in the background of the discussions of daily life. We all are disposed to regard time as endless, and space as boundless. Problems about what lasts "forever," or about what had "no beginning," are suggested to us by familiar considerations. Even children ask questions that imply the insistence and the interest of this conception of infinite time. The adult mind, in our modern days, is reminded constantly afresh of this conception by the facts of geology, and by the theory of evolution. On the other hand, astronomy just as constantly suggests the problem of the boundlessness of the world in space. And theology knows the problem of the Infinite in the form of well-known questions concerning the infinity of God, and concerning what this infinity, if it is admitted, implies. Even if one regards all such problems as insoluble, there remains, for any student of human nature in general, and of the religious consciousness in particular, the question: What are the deeper motives that make man so disposed to conceive both the universe and God as infinite?

Yet the problem of the Infinite, in any of its forms, is so ancient, and has been so often discussed, that anyone who raises it anew has to meet at once the objection that he can only thresh again the old straw. I may as well say at the outset, therefore, that the following paper seems to me to be justified by the fact that certain of the "recent discussions of the concept of the Infinite," to which my title refers, have set these ancient problems in a decidedly new light. This paper is in the main, therefore, a report upon what, in France, has of late been called, in philosophical discussion, the "New Infinite." I myself care little for this modern fashion of recommending ideas merely by prefixing the adjective "new." Truth is never essentially new, being always eternal. But if the adjective "new" serves to make a reader patient enough to attend to one more essay on a topic which Aristotle so skilfully outlined, which the Scholastics so patiently elaborated, and which the modern discussions of Kant's Antimonies may seem to some to have long since exhausted, I will not hesitate to employ the so much abused word. As a fact, recent discussion has put the concept of the Infinite in what, to me, seems a decidedly novel light. We seem to be at the beginning of the attainment of quite unexpected insight as to the logic of all discussions about infinite collections, complexities, and magnitudes. While the discussions to which I refer have been begun, and have been, in the main, carried on by certain mathematicians of a somewhat philosophical turn of mind, they have now reached a point where, as I think, the general students of philosophy and of theology should no longer ignore them. In a recent publication of my own,[1] I have endeavoured in several passages to apply the results of these mathematical students of the logic of the Infinite to the consideration of central metaphysical problems. In the present paper, however, I shall attempt little that is original. I shall be content if what I say serves to indicate to any fellow-student that the problem of the Infinite is as living a problem to-day as it was when Aristotle first attacked it, and that new results, of unlooked for exactitude and clearness, have lately been obtained in this ancient field of work.

[1] *The World and the Individual,* 2 vols., London, 1899 and 1901. See especially the *Supplementary Essay* appended to the first volume.

I.

The scope of the present essay must first be briefly indicated. I have mentioned the fact that some rather mysterious motives, lying very deep in human nature, have led many men to believe that the world is infinite, and to assert that God is infinite. Such beliefs and assertions, in their origin, antedate any clear consciousness, on the part of those who first maintain them, both regarding what these motives for such doctrines may be, and regarding what the very concept of infinity itself means. That this unconsciousness about the meaning and the grounds of our belief in the Infinite does go along with our early assurances about the infinity of things can be shown both by the case of Anaximander, and by that of any thoughtful modern child who asks questions that presuppose an idea of the infinity of the universe. Accordingly, when we try to come to clearer insight about the problem of the Infinite, we naturally have to distinguish two questions. The one is a purely logical question:—What do we mean by the concept of the Infinite? The other is a metaphysical question:—What grounds have we, if we have any grounds, for asserting that the real universe, whether divine or material, whether spatial or temporal, is infinite? The rational answer to the latter question presupposes that the first question has been answered. On the other hand, an answer to the first question might leave the second question wholly open.

Now the present essay will be mainly devoted to the *first* of these two questions. I shall discuss, for the most part, the concept of the Infinite. The question whether the real world, or whether God, is actually infinite, will merely be touched upon as I close. It is the logic and not the metaphysic of the problem of the Infinite that will here form my main topic.

Yet I admit, and in fact insist, that the whole interest of the logical issue thus defined lies in its relation to the metaphysical issue. I am well aware how barren a consideration of the mere concept of the Infinite would be, if it did not help us towards a decision of the problem whether the real world is infinite or not, and nevertheless I feel that, in the present state of philosophical study, we must take the trouble to dwell somewhat carefully upon the merely preliminary problem, even at the risk of being accused of elaborating a mere concept, and of neglecting an appeal to the concrete facts of the real world. For I find, as I look over the history of the problem of the Infinite, that much of the ordinary treatment of the matter has been confined to a certain fatal circle, in which the students of our problem have been led round and round. First, the aforesaid motives, vaguely felt, have forced men to make the hypothesis that the world is infinite. As soon as one has tried to analyse these motives, one has observed that certain aspects of our experience do indeed furnish apparent grounds for believing in the infinity of the universe. But hereupon, becoming critical, one has said: Yet the concept of *what* the Infinite is and means seems to transcend the limits of human intelligence. And so one has refused to consider farther the evidences for the reality of the Infinite, simply because of this supposed incomprehensibility of the conception. On the other hand, any effort to clear up the conception of the Infinite has often met with the objection that a mere analysis of ideas is tedious, and that one wants light as to the facts. Thus, however, the problem of the Infinite has often failed to receive fair treatment from either side. The facts bearing upon the matter are ignored, because the concept is too difficult; and the concept is neglected on the plea that the facts alone can be decisive. I desire anew to break into this fatal circle. Let us make at least our concept of the Infinite clear, and then we shall be prepared to be just to the facts which indicate the infinity of the universe.

In expounding the newer conceptions of the Infinite, I shall follow, as I have already indicated, the lead of certain mathematicians, in particular of Richard Dedekind and George Cantor.[1] I shall use, however, in part, my own illustrations, and shall try to emphasise in my own way the philosophical, as opposed to the mathematical, significance of the ideas in question. I shall then briefly indicate how the new ideas ought, in my opinion, to modify all future discussion of the evidences regarding the actual existence of infinite beings.

I may also say, at once, that my discussion of the concept of the Infinite will have relation not so much to the concept of infinite *magnitudes* (such as is ordinary Euclidean space when it is viewed as possessing volume), but rather to the concept of *collections*, whose units exceed in number the number of any finite collection of units. The conception of an infinite magnitude, such as an infinite volume or an infinite mass, would require for its statement certain conventions regarding the measurement of magnitude, which do not here need our attention. I shall confine myself to defining infinite collections, and infinitely complex systems of objects. We shall see that the metaphysical, and in particular the theological, applications of our concept of the Infinite are especially related to this aspect of our topic, while the conception of an infinite magnitude, in the narrower sense of that term, has less philosophical interest.

II.

In order to help us towards this new conception of the Infinite, let us begin by reminding ourselves of a very simple

[1] A fuller account of the literature than is here possible I have given in the course of the *Supplementary Essay* just cited. The definition of Dedekind is contained in his now classic essay: *Was Sind und Was Sollen die Zahlen?* This paper has recently been translated into English, and published in a volume entitled *Essays on Number*, by the Open Court Company of Chicago. George Cantor's numerous papers are widely scattered. Their substance has been in part summarised in the admirable book by Louis Couturat: *L'Infini Mathématique* (Paris, 1897). A fuller statement of the technical results has lately been given, from the mathematical point of view, by Schönfliess, in his *Bericht über die Mengenlehre,* in the eighth volume of the Proceedings of the *Deutsche Mathematiker-vereinigung.*

observation, which many of us may have made in these days when advertisements are so constantly before our eyes. It has occasionally occurred to some ingenious manufacturer, when in search of a trade-mark, to use, as such a trade-mark, a picture of one of the packages wherein his own manufactured product is put up for sale. Carrying out this plan, the manufacturer in question accordingly puts upon every package of his goods a label whereon is engraved this trade-mark. We can all recall, I fancy, packages of proprietary articles labelled in this way. Some of us may have noticed, however, in passing, a certain logical consequence which this plan involves, if only we suppose the plan rigidly carried out. Each labelled package is to bear upon itself, in a curiously egotistical fashion, a picture of itself. But the package, thus labelled with its own picture, inevitably requires the picture to contain, for accuracy's sake, as precise a representation as is possible of the appearance, not only of the whole package, but of every visible detail thereof. The label, however, itself is a detail belonging to the appearance that the package presents. Accordingly, the picture that constitutes the label must contain, as part of its own detail, a picture of itself. What we see, then, on the actual package, is a picture of this package; while this represented package has upon itself, in the picture, a second trade-mark label, which again contains a picture of the first package, and so once more of the label itself; and this series of pictures within pictures continues before our eyes as far as the patience or the wages of the engraver of the trade-mark have led him to proceed in the work of drawing the required details. Now it may have occurred to some of us that, if the plan of such a trade-mark as this were to be exhaustively carried out, without any failure in the engraver or in the material to hinder its expression, the pictures within pictures, which the plan demands, would soon become invisibly small. In fact, it is not hard to see how, by means of a single definable plan, viz., by means of the one requirement that the package shall bear upon itself, as label, a perfectly accurate pictorial representation of itself, including in this representation the label which the package bears, one logically prescribes an undertaking that could not be exhaustively carried out if the label itself contained only a finite series of pictures within pictures, however long that series might be, or however minute the detail. Just as the label would fail to picture the whole package of which itself is a visible part, unless the label contained a picture of itself, so any picture of the label thus contained within a larger picture of the label, and of the package, would be imperfect unless, however small it might be, it contained a picture of itself; and thus there could be no last member of the series of pictures within pictures, which the one plan of making the label a perfect picture of the package would prescribe.

Now this system of the package, the picture of the package, the picture of this picture, and so on, is a system defined by a single, and in one sense, a very simple plan. We may at once give this plan a name. We shall call it a plan of a particular sort of Self-Representation, a plan whereby a whole is to be pictured or represented by one of its own parts. It is a simple plan, because in order to define it you have only to define:—first, the formal conception of a perfect pictorial representation of an object (a conception which, of course, remains for us an ideal, just as any geometrical definition is an ideal, but which is a perfectly comprehensible ideal); and secondly, the equally formal conception that the picture shall be contained in, or laid upon, the object that is pictured, and shall form a part thereof. Put these two purely formal and perfectly definite ideas together, and the proposed plan is exactly defined.

Let us consider the two ideas for a moment separately. We know what it is to conceive that a visible object, O, shall have a picture, R, which shall precisely represent its every visible detail. In order to form this conception apart from the other one that I just combined with it, we are not obliged to conceive that the picture R is to be as large as the object O. That a smaller picture should still be a perfect representation of a larger object is a perfectly definable ideal. What we mean by this ideal is merely this, that to every variety of detail in the object there shall correspond some precisely similar variety of detail in the picture. Thus, if the object consisted of two lines, arranged in a cross, the picture would simply be another cross. If the object consisted of seven distinct points, arranged in a row, the picture would be a row of seven points. So far there is indeed no requirement that either object or picture should be infinite, or even moderately complex.

And next we may view the other one of our two ideas by itself. That a visible object, R, should be a part of a larger object, this is also a precisely definable idea, and a very simple one. This idea, moreover, is, upon its face, not at all inconsistent with the former idea.

But hereupon, in order to define what we have called the plan of Self-Representation, we have only to suppose these two separately definable ideas, that of the perfect picture, and that of the part contained within and upon the whole, to be combined, so that a visible object should be produced that contained, as a part of itself, a perfect representation of itself. But at once, so soon as, by this combination of two perfectly comprehensible and consistent ideas, we define the plan of self-representation, we observe that no finite degree of complication of object and picture would enable us to conceive the plan perfectly carried out. An object that contained, as part of itself, a perfect picture of itself,—in other words, a self-representative object or system of the type here in question,—would of necessity prove to be an object whose complexity of structure no finite series of details could exhaust; for it would contain a picture of itself, within which there was to be found a picture of this picture, and a picture of this second picture, and so on without end.

III.

The trivial illustration of the nature of a Self-Representative System which we have just used, has thus a deeper meaning than we should at first suppose. We define a comparatively simple plan; but hereupon we come to see that the plan demands, for its complete expression, an infinite series of details. And we see at once that the self-representative character of the plan is the logical ground for this infinity of the required series. The self-representation of a whole by one of its own parts would, if carried out, imply that the whole in question had an infinitely complex constitution. But hereupon let us turn for a moment from this study of the explicitly self-representative systems to the consideration of an object that we all of us are accustomed to regard as at least a possible object of thought, and that we are all disposed to conceive as, at least potentially, an infinitely complex object. I refer to the mathematical object known as the series of whole numbers, 1, 2, 3, 4, 5, and the rest. We all agree that, in our conceptions at least, no whole number that you can name can be regarded as the last of the possible whole numbers. Any series of numbers that we can at present write down, or that we can count in a finite time, will be a finite series. But no such a finite series can exhaust the possible whole numbers. On the other hand, what we mean by the objects called whole numbers is something perfectly precise. The possible whole numbers form no *finite* collection; but they do form a perfectly *definite* collection of objects,—definite in the sense that this collection excludes from its own domain all other objects. We have no difficulty in telling, when any object is brought before our notice, whether it is a whole number or not. Thirty is a whole number; but $\frac{1}{3}$ or $\frac{1}{10}$ is not a whole number. A tree or an angel is not a whole number. Thus the collection of possible objects called whole numbers, although, in one perfectly definite sense, it is a boundless collection, having no last term, is still far from being an all-inclusive collection. It is infinite in one sense; but, in another sense, it is strictly limited and exclusive of whatever lies outside of it. Cantor would call such an infinite collection a *well-defined* collection (*wohldefinirte Menge*) of possible objects,—endless, but in no sense vaguely endless,—since of all possible objects you can exactly say whether they belong to the collection in question or not.

Let us, then, accept for a moment the whole-number-series as a collective object of our thought. Let us regard it as infinite in the merely negative sense of having no last term. I now wish to call attention to an interesting consequence of viewing the number series thus. If you choose, you can, namely, view the whole number series as containing within itself a perfectly definite part of itself, which is, in a precise sense, a complete representation or picture of the whole series. For the series of whole numbers is essentially characterised by the fact that it has a first member, a second member, a third member, and so on without end. Granting this, as the essential character of the series, let us consider a certain perfectly definite portion of the whole number series, namely, the series of even numbers. That series has a first member, 2; a second member, 4; a third member, 6; a fourth member, 8; and so on without end. Now, suppose that under a series of the whole numbers, I write the series of even numbers in order, thus:—

$$1, 2, 3, 4, 5, \ldots .$$
$$2, 4, 6, 8, 10, \ldots .$$

It is plain that, just as I conceive that no number in the upper series is the last of the whole numbers, so I am forced to conceive that no even number in the lower series is the last of the even numbers. It is also plain that, however far I might extend the upper series, by writing in order the whole numbers up to any whole number, n, however large,—I might still extend the series of even numbers by writing them in order up to $2n$. The lower series might thus always remain just as complex and just as well-ordered a series as the whole numbers of the series above it. And thus the lower series would form, as a possible fact, a precise picture of the upper series. Speaking in general terms, I can say that to any whole number n, however large, there always corresponds, in this way of arranging matters, an even number, viz. $2n$, so that the lower series is able to furnish, from its stores of possible members, the resources for the picture or representation of every whole number, however great, and of every series of whole numbers, however long. The world of the possible even numbers is, so far as the possession of a first, a second, a third, and no last member is concerned, precisely as rich as the whole number series. Thus, then, there is an exact sense in which I can say, the complex object called the totality of the even numbers precisely mirrors, depicts, corresponds in complexity to, the complex object called the totality of the whole numbers.

But, on the other hand, the even numbers form merely a part, and a perfectly definite part, of the whole numbers. For from the totality called the collection of the even numbers, all the odd numbers are excluded. Yet this mere part is as rich in its structure as is the whole.

This illustration of the even numbers, viewed as constituting a part of the whole numbers,—but a part which nevertheless can be made to represent precisely the whole,—has been much used in the recent discussions of the "new Infinite." A more striking illustration still is furnished, I think, by another series of whole numbers, selected, according to a definite principle, from amongst the totality of the whole numbers. Let us consider, namely, the series of the integral powers of 2, arranged in their natural order, thus:—

$$2^1, 2^2, 2^3, 2^4, 2^5 \ldots .$$

Now it is plain, at a glance, that this series of the powers of 2 is infinite in *precisely* the sense in which the series of the whole numbers is infinite. For there is a power of 2 to correspond to *every* whole number without exception, since every whole number can be used as an exponent, indicating a power to which 2 can be raised, nor is any whole number possible which cannot be used as such an exponent. Hence the series of the powers of 2, as here arranged in order, precisely corresponds, member for member, to the series of the whole numbers. But, on the other hand, every integral power of 2 is itself a whole number. Thus $2^2=4$; $2^3=8$; and so on without end. And the whole numbers that are powers of 2, taken all together, constitute not only a mere part, but in a very exact sense an extremely *small* part, of the entire collection of the whole numbers. For there are infinitely numerous groups of whole numbers which are *not* powers of 2. Thus, all the whole numbers that are powers of 3, and all the powers of 5, as well as all the powers of 7, or of any other prime number, and, in addition, all the *products* of different prime numbers (*i.e.* all numbers such as 3×7, or 5×11), and finally, all those numbers which are products of powers of different prime numbers (*i.e.* all numbers such as $2^2\times7$, or $5^3\times11^2$) are *excluded* from amongst those whole numbers which are powers of 2. And, nevertheless, that part of the whole numbers which consists of the powers of 2 has a separate member to correspond to every single whole number without exception. In other words, this part, small as it is, is precisely as rich as the whole.

IV.

But let us hereupon look back. As we saw in case of the trade-mark, the system of pictures defined by the one plan of requiring a given object to contain, as a part of itself, a complete representation of itself, would prove to be an infinitely complex system in case we supposed the plan carried out. Or, in brief: any Self-Representative system of the sort that we before defined is, in plan or ideal, infinitely complex. But, as the whole number system has just illustrated for us, the converse of this proposition also holds true. Any system of possible objects that we already recognise as infinite in the negative sense of having no last member, is inevitably such that we can at pleasure discover within it a part which is, in complexity, fully adequate to represent the whole. Thus Infinity and Self-Representation (using the latter term in the special sense above defined) prove to be inseparably connected properties of any system of objects that we can precisely define. If a system is to be self-representative in the foregoing sense, it must be infinite; on the other hand, if somehow we already know it to be infinite, we can prove it to be such that in some (yes, in infinitely numerous) definite ways it is self-representative in the foregoing sense of that term.

In view of these facts, it has occurred to Dedekind to offer, as the definition of what we mean by the infinity of a system or of an object, a formula that we may express in our own way thus:—*An object or a system is infinite if it can be rightly regarded as capable of being precisely represented, in complexity of structure, or in number of constituents, by one of its own parts.*

I have to give this definition first in a form that is not yet ideally exact. Dedekind approaches it, in his essay upon the number concept, in a more abstract and exact fashion. But I have said enough to show, I hope, that in this way of looking at the nature of the infinite, there is something worth following up. And as we have here little space for getting a closer acquaintance with these new aspects of our topic, let us at once remind ourselves of what interest a philosophical student may have in such a view of the infinite.

V.

Any self-representative system, if complete, would be infinite. We approached our recognition of that truth by a trivial instance. But the philosophical student knows of one of his own most central and beautiful problems which the formula now reached sets in a somewhat new light. That problem is the problem of the Self. Whatever our view of the psychology of self-consciousness, or of the mental limitations under which we now are forced to live in this world, we must all of us recognise that one characteristic function of the Self is the *effort* reflectively to know itself. Self-consciousness we never fully get, but we aim at it; it is our ethical as well as our metaphysical goal. Now what would be the conscious state of a being who had attained complete self-consciousness, who reflectively knew precisely what he meant, and did, and was? To such a being we easily ascribe godlike characters. God Himself we often conceive as such a completed Self. If other selves than God are capable of such complete self-consciousness, they are in so far formally similar in nature to the divine. But what our observation of the self-representative systems has shown us is, that in their form, however trivial their content, these systems possess *a structure correspondent to the one that we must ascribe to any ideally complete Self, in so far as it is conceived as self-conscious.* A completely self-conscious being would contain within himself, as a part of his whole consciousness, not, of course, a mere picture, but a complete rational representation of his own nature, and of the whole of this nature. In consequence, as we have now seen, he would be, *ipso facto*, an infinite being. *To define the ideally or formally complete Self, is thus to define the infinite.* Conversely, to define the infinite, is to define an object that inevitably has the formal structure which we must attribute to an ideal Self. The two conceptions are convertible. To question whether the infinite is real, or whether any real being is infinite, is, therefore, simply to ask whether the Self, in its ideal completion, is a concept that stands for any actual entity, or whether, in turn, Reality has the form of the Self. Thus the problem of the infinite becomes central in philosophy in a new sense.

Meanwhile, when once we learn to view the matter thus, the concept of the infinite loses its vagueness, its negative aspect, its appearance of meaning simply what lacks boundary, or has no outlines. The conception of an ideally completed Self may be a hard or even a remote one, but it certainly is not a merely negative, or a vague one. Were you all that as a Self you ideally might be, you would not lose definiteness of outline, or precise character, or distinction from other Selves. Yet, as we now see, you would become, in formal complexity, infinite. Hence, to be thus infinite would not mean to be nothing in particular. Nor would it mean to be everything at once. Nor is this exact concept of the infinite one which we cannot grasp. On the contrary, no concept is more precise; and not many important concepts are simpler. To conceive the true nature of the infinite, we have not to think of its vastness, or even negatively of its endlessness. We have merely to think of its self-representative character.

VI.

But if this new concept is simple and exact, it appears to our common-sense unquestionably paradoxical. For we all early learned a certain so-called axiom, used by Euclid, and very generally regarded as a typical case of a fundamental verity. This is the principle that "the part cannot be equal to the whole," or that "the whole must be greater than the part." Now it may appear to some reader that, in the foregoing statements about the even numbers, and their relation to the whole numbers, and in our illustration of the series of the powers of 2, we seem to have come dangerously near to denying the truth of this axiom in its application to infinite or self-representative systems. This seeming is well founded. As a fact, our definition of infinite systems as self-representative depends upon actually denying that this axiom applies to them. It is quite true that the axiom about part and whole applies to all finite systems and collections. But common-sense, in talking about the vaguely appreciated ideas of the infinite which we all form in connection with the notion of infinite space and endless time, has often expressed, in a more or less halting way, its sense that to infinite systems the axiom in question would somehow fail to apply. Subtract a finite from an infinite magnitude, and the remainder, as we sometimes feel, must be as great as ever. But the newer conception of the infinite depends, not upon such a vague sense of failure to apply the old axiom, but upon defining, in a precise way, that property of infinite systems (namely, their property of being self-representative) which, as a property, ensures that the axiom of whole and part does not apply to these infinite systems. As a fact, it is perfectly possible to investigate many mathematically defined infinite objects and collections in a very precise fashion to see whether or no they are equal. It is possible to define two infinite collections that are unequal to each other. It is possible to define the sort of equality or of inequality that is, in such instances, in question, with as much precision as you can use in defining the equality of two finite numbers. And nevertheless it is possible, while retaining all the definiteness of one's conceptions, to make the whole investigation of infinite magnitudes and collections depend upon asserting that, in their case, the part may equal the whole.

Escape from a bondage to arbitrary axioms is, in fact, a necessary condition of exact thinking upon fundamental topics. When you assume an arbitrary axiom, as, of course, you have a right to do, in any particular investigation, it is still necessary, if you want to think in thoroughgoing fashion, to know that this axiom is arbitrary so long as its opposite is not self-contradictory. Consequently, in considering the range of possibilities, you can always suppose the contradictory of your originally assumed axiom to hold true for some conceived range of at least possible being. Now, the so-called axiom about whole and part comes to us in the first place not as an absolutely necessary presupposition of thought, but as an empirical generalisation, founded on our experience of finite collections and magnitudes. *Why* this axiom holds true for finite collections we do not ordinarily see. It is something to learn that this axiom applies to them precisely *because* they are finite; and that a realm of equally exact and definite objects of thought is possible, to which this axiom does not apply.

Let me try to show the way in which Dedekind, in his essay on number, and Cantor in his theory of the relationships of infinite assemblages of objects, agree, both as to the exact definition of the concept of the equality of two collections of objects, and as to the precise sense in which, in case of infinite collections, a part may be equal to a whole.

What do we mean by calling any two collections of objects numerically equal to each other? The answer is easily suggested by an illustration. Suppose us to know that there is a company of soldiers marching along a street, and that every soldier in this company has a gun upon his shoulder. We need not in this case count how many soldiers there are in the company in order to know, with precision, that there are precisely *as many* guns in that company's equipment as there are soldiers in the company. Here the equality of the two collections is defined in terms of what the mathematicians call a relation of one to one correspondence. By hypothesis, the law holds that to every soldier there corresponds one, and only one, musket, while to every musket in question there corresponds one, and only one, soldier, namely, the man who carries it. To know this law is to know the numerical equality of the two collections. Counting is in this case unnecessary. It makes no difference whether the company contains fifty or two hundred soldiers. In any case, if the supposed law holds true, there are as many guns as soldiers.

With the conception of equality thus illustrated, we are free from the necessity of always counting definable collections of objects before we know whether they are equal. We may then define equality in general thus:—If A and B are two collections of objects, and if a general law is known whereby we are able to be sure that to every individual object in A there corresponds, or may be made to correspond, one object, and one object only, in the collection B, and if the inverse relation holds true, then the two collections A and B, by virtue of this one to one correspondence, are equal to each other.

Now Dedekind, in his mentioned essay, first defines the conception of equality in these terms, and then gives to his definition of an infinite collection a more exact form than we have yet used, by combining this conception with one other equally simple and exact notion. This second notion is that of

Whole and Part. The precise definition of the relation of whole and part, as applied to the case of collections of objects, is as follows: Let there be two collections, A and B. Let it be known, either directly through a definition, or otherwise, that every object which belongs to the collection B, belongs to the collection A, while it is also known that there are objects of the collection A, which do *not* belong to the collection B. Then the collection B is to be called a part of the whole collection A.

Premising these two distinct conceptions, that of equality and that of the relation of whole and part, then Dedekind proceeds to his definition of an infinite collection as follows: A collection is infinite if it can be put in *one to one correspondence, or can thus be found equal to, one of its own parts.* This definition Dedekind introduces, in his essay upon the number-concept, in advance of any definition of the whole numbers themselves. He thus defines the infinity of a collection while using *only the two concepts of the one to one correspondence, and of the whole and part relation.* He thus logically expresses his conception of the infinite quite in advance of stating any definite conception of what a finite collection is; and, in the order of his definitions, tells us what the infinite is, before he shows us how to count three, or ten, or any other finite number.

But, as an objector may here say, mere definitions do not of themselves ensure the possibility of their objects. Can Dedekind show us, apart from mere empirical illustrations of the plausibility of his idea,—can he show us, I say, that a collection defined as infinite in his sense is a possible collection? Is not the very notion a contradictory one? How can the whole be equal to the part?

I answer, Dedekind easily shows that his conception of the infinite *can* be applied without any self-contradiction. Or, as he says, he can show that there are possible systems of objects, infinite in his sense of the term. He names at once such a system. "The realm," he says, "of the totality of my possible thoughts" is, in his exact sense, an infinite realm.

For, as Dedekind continues, to any thought of mine,—let us say to the thought as s, for example, to my thought of *my country*,—there may be made to correspond, in the realm of my *possible* thoughts, *another* thought which we shall call s^1, and which we may suppose to be the thought whose expression would be: "The thought s (viz., the thought of *my country*) is one of my thoughts." If the world of my *possible* thoughts contains the possible thought s, it certainly also contains the *possible* thought s^1. Now let us call all thoughts of the form s^1, *reflective thoughts*. Thoughts of this reflective type are thoughts that consist in thinking, concerning some other possible thought, that "this is one of my thoughts." Now, to *every* possible thought of mine, without exception, there can be made to correspond, in the realm of my possible thoughts, one and only one distinct thought of the form s^1, and *vice versa*. Hence, the whole collection of my possible thoughts, and the collection of the possible thoughts of the type s^1, *i.e.* of the reflective thoughts, are precisely equal, just as the two collections of the muskets and of the soldiers are equal. For the two collections of thoughts correspond to each other, in one to one fashion, precisely as the guns correspond to the soldiers. Yet the collection s^1 is a perfectly definite part, *and is not the whole* of the realm of my possible thoughts. For there are thoughts, such as the simple thought of *my country*, which are not reflective. In this realm of my possible thoughts, a part may, therefore, be equal to the whole, not vaguely, but in a perfectly definable fashion. Hence, by the definition, this realm or collection of the totality of my possible thoughts is infinite. Yet surely the conception of the realm of all my possible thoughts is not a contradictory conception.

VII.

Thus, then, the logical basis for the new concept of the Infinite is, in its outlines, complete. One can define infinite collections without making use, in the definition, of their merely negative character of being *without* end. One can define them by telling what they *are*, rather than what they are not. One can form a basis for distinguishing such collections, in a definite fashion, both from one another, and from all finite collections. One can, consequently, name a criterion upon which to base arguments regarding the question whether infinite collections exist in the real world. For the question as to the real existence of infinite collections *becomes identical with the problem whether the real world contains facts, or systems of facts, which possess a certain sort of self-representative structure.* Or, in other words, the problem of the reality of the Infinite becomes identical with the problem whether the universe, or any portion of the universe, has the same form or type which we are obliged to attribute to an ideally completed Self.

Whatever considerations make for an idealistic interpretation of reality, thus become considerations which also tend to prove that the universe is an infinitely complex reality, or that a certain infinite system of facts is real. For Idealism, in defining the Being of things as necessarily involving their *existence for some form of knowledge*, is committed to the thesis that whatever is, is *ipso facto* known (*e.g.* to the Absolute). But the knowledge of any fact, if this knowledge exists at all, is itself a fact. Hence the essence of Idealism lies in its thesis that *to every fact corresponds the knowledge of that fact*, while every act of knowledge itself belongs to the world of facts. Since, however, the fact-world, even for Idealism, contains many aspects (such as the aspects called feeling, will, worth, and the like), which are *not* identical with knowledge, although, for an idealist, they all exist as known aspects of the world, it follows that, for an idealist, the facts which constitute the existence of knowledge are themselves but a part, and are not the whole of the world of facts. Yet, by hypothesis, this part, since it contains acts of knowledge corresponding to every real fact, is adequate to the whole, or, in Dedekind's sense, is equal to the whole. Hence the idealist's system of facts must, by Dedekind's definition, be infinite. Or—in brief—for the idealist, the real world is a self-representative system, and is therefore infinite. But I have myself also endeavoured to show, in my *Supplementary Essay* already cited, that a similar consequence holds for *any* metaphysical system, even if such a system is not idealistic. For, as I have there attempted to explain at length, every metaphysical interpretation of the universe, whatever its character, must imply that the real world is a self-representative, and is consequently an infinite system. In consequence I conceive that Dedekind's definition of the Infinite leads us to the important result of being able for the first time to show explicitly that the real world, whatever else it is, is an infinitely complex system of facts.

The ancient objections to supposing anything real to be infinite in its complexity of structure, the time-honoured arguments against asserting that the infinite is real, have all rested, in the end, upon the supposed *indeterminateness* of the concept of an infinite collection, or of the infinite in general. But the exact definition of Dedekind enables us to conceive the Infinite, in any one of its special instances, as something perfectly precise and determinate. For instance, let us suppose the collection of *all the whole numbers* to exist as a fact in the world. This collection has positive properties, which, as Dedekind has shown, follow necessarily from his definition of the infinity of the collection. Now this collection contains a part, precisely equal in complexity to the whole, namely, the collection before mentioned, of all the powers of 2.

Now, although this part of the whole collection of the whole numbers is an infinite part, whose infinity can also be defined in Dedekind's positive terms, yet it nevertheless is a perfectly determinate part. For, if we ask what whole numbers are *left over*, when, from the infinite collection of all of them, taken together, we remove or subtract the entire infinite collection, or part, called the powers of 2, the answer is perfectly definite. For the whole numbers that are *not* powers of two, themselves form a precisely definable collection. We can even go much further. From the infinite collection of the whole numbers we can suppose subtracted or removed, in succession, an infinite series of collections of whole numbers, *each* of which collections is infinite; and yet, if the process is exactly defined, we can tell precisely what will be left over *after* all this infinite series of subtractions is carried out. For, to exemplify this fact:—the prime numbers, 2, 3, 5, 7, etc., form of themselves a demonstrably endless series of whole numbers. For there is no last prime number. Now let us suppose that from the collection consisting of all the whole numbers, we first remove or subtract the infinite collection of *all the prime numbers.* Suppose that we *next* remove the infinite collection of *all the squares* of all the prime numbers. Then let us remove the infinite collection of *all the cubes* of the prime numbers; then all the *fourth powers* of all the prime numbers. Let us continue this process without end, each time removing an infinity of whole numbers, but continuing to infinity the process of removing higher and higher powers of each prime number. Will the final result of this entire infinite series of successive subtractions of infinite parts from the originally infinite whole be in the least indeterminate? On the contrary, we know at once what whole numbers will survive the process. For the numbers that will remain over *after* the completion of the infinite series of removals will be those whole numbers which are either the products of different primes, or else the products of powers of different primes. Thus precise may be the results of reckoning with infinite collections, if only we use the right, which Dedekind's view of the positive infinite gives us, to regard every such collection, as soon as it is precisely defined, as an actually possible and given totality, with precise relations to other totalities, finite and infinite.

Nor are such elementary instances of the possible exactness of our conceptions of infinite processes by any means the principal examples of the essential determinateness of the infinite. Cantor, whose researches have wrought such a revolution in our knowledge of infinite collections, has been able to show that, despite the wonderful plasticity which the foregoing concept of the equality of two infinite collections obviously possesses, the concept, as defined above, nevertheless has an exactly limited range of application. For there are definable collections, infinite in the foregoing sense, which are demonstrably *not* equal to one another. That is, there are cases where we can show that, of two given infinite collections, one so exceeds in complexity the other that a one to one correspondence *cannot* be established between them. In such a case, one of the two collections may indeed be a part of the other, but will then be, in this case, a part which although infinite, is *not* equal to the whole. Our previous definition of the infinite, in fact, while it depended upon pointing out that, in infinite collections, the part *may* equal the whole, did not assert that an infinite collection must be equal to *every* one of its own parts, but asserted only that an infinite collection is equal to *some* of its parts. In case, however, an infinite collection contains certain infinite parts to which it is not equal, but which it exceeds in such fashion that a one to one correspondence between the whole and such a part is impossible, then the greater infinite collection is said by Cantor to be higher in *Mächtigkeit* or in Dignity than is such a lesser part. The concept of the Dignities of the infinite, which Cantor thus introduced, depends upon proving that precisely such gradations of infinity are to be found in case of certain definable collections of possible objects. As a fact, it can be shown that the collection consisting of *all* the possible fractions, *rational and irrational*, between 0 and 1, is of *higher* dignity than is the collection of all the whole numbers. On the other hand, a collection consisting merely of all the *rational* fractions, is of the *same* dignity as is the collection of all the whole numbers. The proof of both these results can be given in a perfectly elementary form, which is indeed too lengthy to be stated here, but which can be made comprehensible to almost any careful student who retains the slightest knowledge of elementary arithmetic and algebra. Yet the first discovery of these Dignities or gradations of the infinite, as made by Cantor, constitutes one of the most ingenious advances of recent exact thinking. Cantor himself has shown (and independently Mr Charles S. Peirce has done the same), that there is an endless series of these possible Dignities of the infinite.

The result of such researches is, however, to show in a new way how determinate an object an infinite collection, once exactly defined, proves to be. For an infinite collection of a lower Dignity, although unquestionably boundless in its own grade, remains in a perfectly definite sense incomparably small when considered with reference to an infinite collection of a higher Dignity. Infinity, and precise limitation, are thus shown to be perfectly compatible characters. For no process of numerical multiplication, even pursued *ad infinitum*, can directly carry one from an infinity of any lower Dignity to one of a higher Dignity. The transition from one grade to a higher grade can be made only by means of certain precisely definable operations which are not expressible in merely quantitative terms. The lower and the higher Dignities are thus separated by logically sharp boundaries of which earlier speculations upon the infinite gave not the slightest hint. But these boundaries, existing in the realm of what was once the "void and formless infinite," show us that henceforth no one who identifies the infinite with the indeterminate is aware whereof he speaks; and that no one who conceives the infinite merely in terms of the negative "endless process" can be regarded as having grasped the true nature of the problem of the infinite.

Meanwhile, to look in yet another direction, the concept that, in an infinite system, the part *can*, in infinities of the same Dignity, be equal to the whole, throws a wholly new light upon the possible relations of equality which, in a perfected state, might exist between what we now call an Individual, or a Created Self, and God, as the Absolute Self. Perhaps a being, who in one sense appeared infinitely *less* than God, or who at all events was but one of an infinite number of parts *within* the divine whole, might nevertheless justly count it not robbery to be equal to God, if only this partial being, by virtue of an immortal life, or of a perfected process of self-attainment, received, in the universe, somewhere an infinite expression. The possible value of such a conception for theology seems to make it deserving of a somewhat careful attention.

I conclude, then, by urging the concept of the "New Infinite" upon the attention of students of deeper theological problems. I believe it to be demonstrable that the infinite is, in general, neither something indeterminate, nor something definable only in negative terms, nor something incomprehensible. I believe it to be demonstrable that the real universe is an exactly determinate but actually infinite system, whose structure is that revealed to us in Self-Consciousness. And I believe that the newer researches regarding the infinite have set this truth in a new and welcome light.

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