Josiah Royce

The Columbia Lectures of 1904

Lecture III

Archives copies made by Scott Pratt
First Transcription by Jon LaRochelle and Paula Wright

Numbers in square brackets are the page numbers of the manuscript or original typescript.

Lecture 3.

[21] The somewhat abstract considerations with which the last lecture closed must be considered for a moment in a more practical light before we go on to apply them to the study of the more complicated scientific concepts. When I speak of symmetrical and non-symmetrical, of transitive and intransitive, relations, I make use of terms that may seem painfully or uselessly abstract, but which are nevertheless necessary for the process of definition. But on the other hand I do not wish to neglect the fact that the distinctions amongst the relationships which are thus defined have a very decided concrete and practical significance. We all recognize in daily life that our symmetrical relations have a peculiar value, and also lack certain values which are possessed by our non-symmetrical relations. I stand in a symmetrical relation with my fellow when I am his brother, his friend, his equal, his fellow-traveler, when co-workers in an activity we are both symmetrically related. On the other hand, I stand in a non-symmetrical relation to my creditor or debtor, to my ancestors and to my children, to whoever is better than I am or whoever I regard as worse than I am. The symmetrical relations have in many cases a pleasing and simple and satisfactory aspect. They do no arouse hostilities, they do not of themselves invite change. They have, on the other hand, very distinctly what may be called from a practical point of view the levelling character. And what brings life to a level is not the most interesting aspect of life. In so far as I stand on the same level there may be, indeed, many observable differences amongst us, for difference is itself a symmetrical relation, but on the other hand these differences are not viewed as exciting us to definite and serial forms of activity. To attain or to establish equality [22] amongst men has often been an object of the most vehement conflict, so long as the equality was not attained; but when equality is attained, it gives us merely the opportunity for some type of action, it does not of itself put us into significant courses of activity. The fact that I am equal to my fellow does not determine why I should select one profession and he another. It does not tell me which of us ought to write a letter to the other, or

which ought to greet the other first. Hence mere equality is always unsatisfactory. And I no sooner make men equal than they begin to discover new devices for establishing inequality among them.

On the other hand the unsymmetrical relations are precisely those which most have to do with inciting changes, with leading to coherent activities, and as we shall soon see, with establishing precise serial order amongst men and amongst activities. It takes but a glance at the facts to show you that in terms of before and after, of right and left, of up and down, of to and fro, of advance and retreat, of ascent and descent, - in terms of such relation you set in order your house, arrange your business, and establish your plan of life. In nature the non-symmetrical relations appear to us even the sole ones upon which life, change, growth, decay, and the course of nature generally, depend. The relation, a greater temperature and the less, is a non-symmetrical relation. But heat passes from the body at the higher temperature to a body at the lower temperature, and does so itself. And the relation between a high level and low level is a non-symmetrical relation. Water flows from the high level to the low level. The relation between infancy and maturity is a non-symmetrical relation. The infant tends to mature. Thus the distinction between symmetrical and [Pages missing; resuming on page 25] relation, there can be no doubt of the importance of such transitivity. In general, where one object can be substituted for another, unless only a single substitution is from the nature of the case permitted, substitution of relations becomes the intransitive warrant of a large number of very significant real or ideal transformations, with which in another connection we shall soon become better acquainted. And so transitivity has a great deal of practical significance. On the other hand, intransitive relations are of great service for certain purposes of defining some groups of objects, of keeping objects and groups of objects apart, and of setting limits to certain processes.

We see from this very general survey that the conception of symmetrical has a close connection with the idea of a level. We also see that the conception of a non-symmetrical relation has a close connection with the idea of a transformation and with the constitution of an ordered series. We also see that certain types of connection depend upon the establishment of transitive relations, we further see that intransitive relations may be significant as breaking off or limiting certain connections. It is also true that we may use intransitive relations as the basis for the definition of transitive relation, as we shall erelong also exemplify.

I turn from the abstract definition of relations to the application of what we have learned to the definition of the nature of an ordered series. We saw, when we first mentioned the ordered series, of what very vast range and various character the types of ordered series are. It is important to point out at once that however various ordered series may be, they are all variations upon a single [26] axiom, an axiom whose statement is so simple that one wonders why the generalization in question was not sooner made than it was. As a fact, the generalization that I am about to mention began to become prominent in the literature of exact science some twenty-five years since. A remarkable paper by Benjamin I. Gilman in the "English Journal of Mines" in the year 1890 illustrates the results that up to that time had been reached regarding the nature of a one-dimensional ordered series and adds the author's new interpretation of these results. Quite recently Russell in his "Principles of Mathematics" has elaborately discussed the whole subject.

Let me state at once what the simple axiom is of which all types of ordered series may be regarded as variations. Let us consider a one-dimensional ordered series, that is, a row of objects such as might be conceived to be arranged along one straight line. This series might be what is technically called continuous, as the series of ideal points on a line is generally said to be. Or this series might be discrete, as the series of whole numbers is. Or it might have various other forms of discreteness into whose variety I cannot here go. It might be a series made up of operations or stretches that were continuous, and of operations or stretches that were in any way you please discontinuous. In any case, in so far as the series could be regarded from any point of view as a single, one-dimensional series, it would have these characters: - (1) A11 the elements that were found in the series might be said to form a class of objects, so that the ordered series is in the first place a particular kind of class. (2) A relation can be found, we will call it the relation "R"; it is a relation such that if you take any two objects of a series, "a" and "b", then "a" stands to "b" in this relation "R". The relation "R" is non-symmetrical [27] and transitive. It will follow from these conditions, and from these conditions alone, that the class of objects in question must form a single, one-dimensional series.

This abstract definition may be made clearer by considering the concrete case of a row of men arranged along a straight line. Suppose that one end of the line is taken as the head and the other end of the line as the rear of the row of men, so that if a man is nearer the head, then another ahead of him is said to be in front of him, or if nearer the rear is said to be behind him. Then of this row of men it is of course true that there is a relation "R". Here the relation of "in front of." If this relation is defined so that it is not confined to a case where one man is immediately in front of another but extends transitively all along the line, then I can of course say that of any two men in the line one of the two is in front of the other. But this relation is an unsymmetrical relation, and by definition it is transitive. That is, if "a" is in front of "b" and "b" is in front of "c", "a" is in front of "c". It is evident that all the relations of the row of men will necessarily constitute them a row having the just defined characteristics which follow from the statement, that every man does stand to every other there in this relation or in its converse, so the relation is non-symmetrical and transitive. What is more or less intuitively evident when we observe the row of men, can be put into a logical form that enables us to apply the general law of this series to any other ordered series. No matter how long the row of men is, suppose that we begin by considering two of these men, whom we will call "i" and "j". By hypothesis either "i" is in [28] front of "j" or "j" is in front of "i". Let us consider hereupon any other man "k". Then let us suppose that "i" is in front of "j". Then either "k" is in front of “j” or “k” is behind “j”. If “k” is behind “j”, then since the relation is transitive, “k” is behind “i” and the relations of these three men are settled. But if “k” is in front of “j”, then the place where “k” is to be found is determined by asking the further question whether he is in front of “i” or not. In other words, he is either in front of both “i” and “j”, or he is between them, or he is behind them both. Introduce any other man and ask about his place in the row. And answering the question as to whether or no he is in front of a certain number of men will finally determine his place in the row. Even if we supposed the number of men whom we had to consider was practically or absolutely unlimited, still the one statement which we have already made concerning the relation of “in front of” or “behind” would enable us, if we only had a law for answering it for any once given number of men, to arrange the system so that the place of any member would become as definite and determinate as we chose to make it. For, given any man in the row, all the rest of the men are divided into two classes, those who are in front of him and those who are behind him. If the row has one first member who comes in front of everybody else, then for that member one of these classes is vacant and the other men are behind this man. If the row is limited at the other end, then the last man has everybody in front of him. For all the intermediate members, or for every man in the row in case we supposed the row to be of unlimited length in both directions, the determinate place is found, first by asking the question with regard to any member, “To what class does this [29] member belong with reference to any other member?”, that is, to the class of those who are in front of that member or to the class of those who are behind that member. If we take any two members of the series, then, every other member is either before them both or behind them both or between them, and in this way we can assign the due place to every member of the row, if we only suppose we have some principle by which we can determine, every time we consider a pair of members, where they stand in the fundamental relation “R” to the other members.

A good example of the application of this definition of a series to a series which is actually unlimited, we find in the case of a series which shall consist of all the rational fractions that we can define which are greater than zero and less than unity and which are supposed to be arranged in the order of what is usually called their value. In the ideal row or series of the rational fractions in question, there is no first member and there is no last member. For there is no least one amongst the rational fractions greater than zero, and there is no least one amongst the rational fractions greater than zero, and there is no greatest one among the rational fractions less than unity. On the other hand, you know precisely where the fraction “¾” comes in the series, for you can determine by a perfectly simple and universal law whether “¾” or “⅘” or “11/17” or any other rational fraction we have is greater or less. Meanwhile, if you exclude the admission to the series of different fractions having the same value, by regarding all equivalent fractions as the same fraction, then any two rational fractions if reduced to their lowest terms stand in what is now the relation “R”’; that is, if one of them is greater, the other is less. Thus, although the collection of rational fractions between [30] zero and unity forms an inexhaustible collection, its serial order and the place of every member in the series is perfectly determinate.

It is thus seen that the conception of serial order depends entirely upon the conception of an unsymmetrical and transitive relation. This again, abstract as the statement seems, is a truth that has its constant practical application. If I can arrange sets, or property, or things in series, if I can order the cards in the card catalogue or the names in a directory, or can in any other way arrange objects serially, the whole practical value of the series for my activities depends upon the unsymmetrical and transitive character of the connections involved. So far as the relations are unsymmetrical, there is a distinction, and a practically very significant distinction, between the two directions in which I can traverse the series. On the other hand, so far as the relations are transitive, I have power to predict the characteristics of a given member of the series, by what Professor James has called the “action of skipped intermediaries”. Thus in rapidly going through a dictionary one does not find one’s way from one word to another by carefully reading all the intermediate words, but by using the transitive character of the relations of an alphabetical order and by thus rapidly passing over intermediate stages, sure that if one has not yet passed the point in the series that one was to reach, that point is still to come, so long as one is proceeding in the right direction, while in so far as one has passed already certain points in the series, one will not meet them again, so long as one continues going in the same direction in the one-dimensional series. [31]

There is no time here to speak of the way in which one-dimensional series, as thus far defined, could be so modified as to make possible these serial forms in which a series returns into itself and becomes circular or periodic. Rhythms and other periodic processes have relations to the one-dimensional series just defined which are themselves very interesting, but it is not into the variety of types of series that I here intend to go. What is important for us to note is that the serial order depends upon a character, which we can identify in the operations of our will, as well as in the order of external experience, where this character seems to be of the most universal application to the world, to science, and to the practical arts. The character in question is that in case of a serial order one is dealing with the relations of the type of the relation of means to an end, of preliminary stages to their results, of earlier activities to later activities. One is dealing also with relations of the types that are so characteristic of numbers, of quantities, of magnitudes, of directions in space, of processes in time. It will also be noted that the highly important relations which are both transitive and symmetrical, do not appear as helping us to define ordered series. If you know of a collection of objects that they stand in transitive but symmetrical relationships, that does not enable you to arrange them in any series. For instance, to know that a large number of things coexist does not help you to know in what order they are arranged, but coexistence is a symmetrical, transitive relation. To know that a collection of quantities are equal, again does not help you to understand any serial order. Merely to assert the equality of men establishes no serial order; and this simple logical consideration [32] lies at the bottom of the very conditions which have so frequently connected the indiscriminate assertion of equality with an approach to social anarchy. A world where all the relations that attract our attention were symmetrical and transitive relations, would be chaos. So much, then, for ordered series.

I pass at once to the contrasting type of concepts whose connection with the series we have already observed to be so important. In my former discussion I defined what we called “levels”. A “level”, in the light of our previous illustrations, and of our later and more abstract definitions, may now be defined as a collection of elements such that any two of them stand in a certain relation “R”, which is transitive, like the serial relation is not. The objects that lie upon a given level, are for that very reason not ordered, in so far as you merely know this relation to be existent amongst the members of the various pairs that you can define in its collection of objects that are on the level. Of course, these objects may be on a level with reference to some symmetrical transitive relation. On the other hand, they may have at the same time unsymmetrical and transitive relations which set them into an order. Thus the cross-section of a specimen of rock may have an orderly structure, which is defined in terms of various series. But these various series will be either one-dimensional series characterized by unsymmetrical relations, or they will, if various dimensions are concerned, consist of interwoven ordinal series of [33] various types. But so far as objects observable in the cross-section are simply upon the same level, that fact does not order them. The consequence is that that sort of knowledge that we get of objects that are upon the same level is in so far forth a kind of level knowledge which enables us to identify these objects in various ways, to substitute one for another, to state in certain cases very important laws that hold true for all these objects. But, on the other hand, the kind of knowledge that we get when we study serial order is obtained in so far as we merely put objects on the same level. The discovery or definition of a level may be of course the beginning of some new process of the definition of serial order; as the setting of all men on the same level, as equal members of society, may be the beginning of the establishment of a new arrangement of their activities, or of their personal decay. The contrast between the logical significance of a level and the logical significance of a series is very notably brought out in the modern theory of energy. The modern theory of energy, as you all know, rests upon two fundamental propositions. The one of these propositions relates to an ordered series of events, the other of these propositions relates to the establishment of a certain level. The principle of the conservation of energy is a principle due to the taking of a level. It is the principle that when a system is transformed, so long as this is a closed system, the energy of that system remains in quantity unchanged. That is, whatever the transformations the system undergoes, the level of its total energy does not alter. On the other hand, the other proposition of the modern theory of energy concerns the question, "What transformations will take place in a given system [34] in which the energy is distributed in a given way?" Suppose a distribution of temperature such that the temperatures of the different parts of a body, or of a field of any kind where your temperatures can be found, are different. Then the re-distributions of heat energy will follow certain serial orders, which are determined by the normal differences of temperature. From the principle of the conservation of energy it is quite impossible to get these other principles which refer to the serial order of the events of the world. In that very remarkable survey of the whole theory of energy, which is contained in the brilliant lectures of Ostwald on "Naturphilosophie", attention is drawn in novel ways to the significance of this well-known contrast between the two principles of the theory of energy. The one of these principles, the level principle, the principle of conservation, determines certain conditions under which alone redistributions of energy can occur. The other principle, the so-called "Second law," determines as Ostwald states the case what happens in a given system. But

what happens in a given system must take place in a certain serial order; and in this serial order the relations are also more transitive and non-symmetrical. The vast range of facts which the modern theory of energy covers suggests at once that a logical formula which expresses the difference between the two principles of the theory of energy, has at any rate a very wide range of application. But the principle here involved is really very much wider than the theory of energy. For it applies wherever we have any occasion to consider a level of any type. In a wholly different field from that which the modern theory of energy covers, you have the contrast between the serial order and the levelling principle well [35]
illustrated by the contrast between the sequence of events which constitute the conduct of a given business, say the conduct of a banking business, and that view of the state of the business which the balance sheet prepared to represent any one state of the business, reports. The estimate of the state of the finances of a corporation, or of any business, which can be founded upon the balance sheet, is of course an important estimate. But if it becomes necessary to examine into the prospects of the business, it will be necessary to take into account in addition to the report of the balance sheets, the origin, the conduct, and the result of various enterprises, the source of various obligations and assets which the balance sheet alone cannot enable one to know. If an auditor, a bank examiner, or an inquiring committee of share-holders desire to know completely the whole state of the business, they must consider what has been done and what is planned for the future, where certain paper has its origin, and when certain obligations are to be met in the future, and all such computations belong to the world of serial order. As was shown by our first illustration of the concept of a level, the process of “taking a level” involves in general the correlation of many different series. A level consists of the corresponding members of different series that cross one another in various ways. Hence the kind of light upon the structure of a whole system or upon the nature of a course of events which a level throws, is wholly different from the kind of light which a stated serial order, by

itself alone, disposes us to get. One who confined his attention solely to the attention of serial order, would be unable in any definite way to combine together the results of the different lines of inquiry to be carried [36] on. Or if we undertook to combine such results we could do so only by means of certain types of transformation, whereby we could pass from one series to another. But thus we would only complicate our field by introducing new series, in addition to the old ones. So one who is obliged to explore a woodland by following in detail path after path, observes a numerous series of facts and may be able, if he has sufficient imagination, to interweave these series more or less so that in his mind they constitute a system. But the one who takes a birds-eye view of the country that he has passed over, is in the position of one who considers a large number of facts as coexistent or as in some other respect upon a level. The process called "taking a level" in surveying has the same general significance that the process of taking a level in any conceptual activity possesses, only that the surveyor's line of levels shows us out of various levels a new serial order can be woven, in such wise that new results follow. And as a fact, of course, the processes of ordering series and of taking levels can be and are combined in the most varied fashions. In projective geometry, the process of taking the projection of a given figure by drawing rays from the points of projection upward is a process of establishing certain series. The process of cutting all these rays by a plane upon which an object is projected is a process of taking a level.

I have already mentioned the significance which the conceptual form called an equation possesses in all kinds of science. It is easy for a superficial observer to suppose that the one interest of exact science is in establishing equalities. But a glance at the nature of the number series, or at the interest of science in constructions, in drawing lines, in producing various [37] lines until they meet, in following a sequence of events, in describing a motion, a path, a change in the distribution of energy, or anything of the sort, shows that exact science is quite as much interested in serial order as in any other concept. The equations of mathematics drive their whole significance from the fact that they are levels drawn across a collected of correlated series, of magnitudes, quantities, numbers, and objects symbolized in terms of these.

The whole significance, however, of series and levels, can only be appreciated when we consider their relation to that type of concept which we have not yet studied, namely to the transformations. In giving my general definition of this type of conceptual forms I mentioned the fact that external experience forces upon us upon every side changes of things, that our practical interest lies in directing the course of our experience by means of artificial changes, produced at will; and finally, that for the purposes of our conception we frequently define in our sciences ideal changes which do not directly correspond to any events in the world of our outward experience, but which do enable us to make new combinations, and valuable substitutions of one concept or set of concepts for another. A transformation, or change, viewed not as a fact in the world but as a conceptual form, is a defined or conceived process by which something comes to take the place of something else. In the first place, we observe the actual changes of the world. So soon as in any way generalizing the fashions of change that we observe, we state an empirical rule, such as that Summer follows winter, or night day, we forthwith conceive of a [38] type of change, or we use the conceptual form, or concept of a transformation, as our own internal fashion of thinking how it is that the events happen. As soon as our definition becomes at all exact, and we have a notion of some law that the changes of the world follow, we deal with a transformation that now has in addition to its interest as a report about our facts, a logical character. For instead of merely reporting what happens in the world, we are engaged in describing the way in which we depict what happens in the world by voluntarily letting one of our ideas pass over into another, in accordance with a rule. Popular science, from the very outset has therefore used the conceptual form which we now have in mind in its effort to describe what goes on in this world. When the early Greek physicists defined the process of the world by saying that things in nature follow sometimes the way upward and sometimes the way downward, or that as Empedocles put it, love and hate are striving together in the world for the control and arrangement of the four elements, love tending to mix and hate tending to part them, such great generalizations made use of the conceptual form of a type of transformations. In the conceived world of Empedocles now hate and now love conquers. That is, a certain type of transformation now takes place and now gives place to its opposite.

But when we are engaged not merely in reporting what happens in the world, but in planning what shall happen, our definition of a type of transformation becomes still more the expression of our own interests in things, and less the mere report of the observed sequence of phenomena. In imagination, even when we do not hope to carry out our placs [sic], we transform our lives in accordance with [39] the plans, dreaming of a type of success that we have not attained, perhaps do not hope to attain. We are dealing not with vague imaginations, but with definite ideals; we conceive, perhaps, of a transformed social order, towards the attainment of which we believe that present political activities should be directed. In all such cases we conceive a type of transformation.

The exact sciences, however, furnish us instances of transformations which are not primarily intended either to be reports of what goes on in the world, or to be plans for the guidance of our practical activities in the outer world. These transformations may be more or less deliberate invention of our own thought. Such inventions are generally suggested by experience, but they may come to be quite remotely connected with our present knowledge of natural processes, or even with the interests of our practical arts. A transformation of this kind may be called an ideal operation. One very good example of an ideal operation in mathematics is furnished us by the ordinary operation of addition. It is true that addition in arithmetic and in the computing part of applied science is an operation devised so that it possesses a very limited relation to certain operations that go on in the natural world. But the mathematician does not depend for his definition of addition upon waiting to find out whether nature furnishes him a transformation to which his ideal operation confirms in its type. The transformations of the mathematician when he deals with the definition of addition, are defined by a rule of his own. Given two numbers “a” and “b”, there is a rule that enables you to find out some their number “c”, such that it may be called according to this rule the sum of “a” and “b”. When [40] you add “a” and “b” you transform “a” and “b”, taken as separate numbers, into their sum “c”. That is, you learn for a certain purpose to substitute their sum for them. Projection, which was mentioned a moment ago, as an instance of a geometrical operation, is an example of a transformation. The types of projection upon which the various kinds of maps depend, are given transformations, which are suggested indeed by relations in the outer world, but which are definable in purely mathematical terms as ideal operations.

But in addition to the countless transformations which characterize mathematical science, the concept of a transformation applies as well to the moral world, and to the world of art, as to the world of scientific investigation. A definition of the moral law takes the form of telling one what to do under certain circumstances. As the statement of the moral law involves the definition of an ideal, the moral operations, however much one hopes to see them carried out in practice are defined in a conceptual realm. The precepts of a moralist are statements of ideal transformations. In the same way in art the interest of the artist or of the critic may, and frequently does, lie in the occurrence or in the production of certain changes in a series of actions, or experiences. The theory of music is concerned with certain transformations, such as constitute a musical sequence. The canons of dramatic art, relate to forcible transformations, which occur in the world of ideal actions and which may be presented on the stage. The concept of a transformation, is therefore of universal significance in every kind of thoughtful undertaking. Our present interest lies in indicating how this conceptual form is related to the ones already characterized. [41]

When we conceive a transformation there is, we have said, some state of things which in our minds we allow or require to pass over into some other state of things; and in terms of this transition we define, conceive, or imagine, the transformation that we are talking about. What is essential to a transformation is, then, that some one earlier state of things should at the very last give way to some one later state of things. A transformation, consequently, implies at least one case of a dyadic relation. This dyadic relation is in a very great number of cases non-symmetrical, since we are commonly interested in the transition from the earlier stage to the later stage. But there are a great many cases where our interest in a transformation is so closely associated with our interest in reversing it and in getting back to the point from which we started, that the relations which we manifest in defining a transformation may be symmetrical. Rhythm or periodic processes of a type now several times mentioned, involve transformations with a certain return to earlier stages. And this return involves symmetrical relations, which are usually combined with other relations which are non-symmetrical. In brief, a great many transformations are either reversible, or else in the course of a series of transformations lead to the return of something that was present earlier in the series, so that the whole series becomes in some sense closed. It is evident that the types of relations involved in a collection of transformation may therefore include all the types with which we have thus far become acquainted. Certain transformations may be unique, as the death of a given man is a unique event in his life, and as certain other notable events of life have obviously a unique character. When we [42] conceive such transformations we do so with the use of conceptual forms that imply their intransitive relations, or the inclusion of something that prevents more than a single case of the dyadic relation in question from taking place. But in a large number of other cases a set of real or of ideal transformations may have a serial character. In such cases some transitive relation appears. On the other hand, a large number of transformations may possess a character which leads through all that happens to the repetition of certain transitive but symmetrical relations. In a case like this, the result of the transformation is that everything concerned remains on a given level. And as we already said in speaking of the doctrine of the conservation of energy, our conception of this conservation is that all possible transformations of energy leave its total in a closed system unchanged. Where a set of transformations leaves objects in certain aspects upon the same level, we have presented to us the conception of what is called in the exact sciences the “invariant” of a set of transformations. The conception of an invariant is of the utmost importance in every region where we are able to define the laws of phenomena at all. Our survey of the system of the fundamental concepts of science has thus been strengthened by some consideration of the relations that appear to be involved, by a wider range of illustration than was possible when we first brought these conceptions to your notice. A few further examples at the next time will be needed to give us a sufficient insight into the wide range of application which these fundamental concepts possess.